Noised-induced phase transition in an oscillatory system with dynamical traps

I. Lubashevsky[1,a], M. Hajimahmoodzadeh[2], A. Katsnelson[2], and P. Wagner[3]

[1] Theory Department, General Physics Institute, Russian Academy of Sciences, Vavilov Str. 38, Moscow, 119991 Russia
[2] Faculty of Physics, M.V. Lomonosov Moscow State University, Moscow 119992, Russia
[3] Institute of Transport Research, German Aerospace Center (DLR), Rutherfordstrasse 2, 12489 Berlin, Germany

Abstract. A new type of noised induced phase transitions is proposed. It occurs in noisy systems with dynamical traps. Dynamical traps are regions in the phase space where the regular "forces" are depressed substantially. By way of an example, a simple oscillatory system $\{x, v = \dot{x}\}$ with additive white noise is considered and its dynamics is analyzed numerically. The dynamical trap region is assumed to be located near the x-axis where the "velocity" v of the system becomes sufficiently low. The meaning of this assumption is discussed. The observed phase transition is caused by the asymmetry in the residence time distribution in the vicinity of zero value "velocity". This asymmetry is due to a cooperative effect of the random Langevin "force" in the trap region and the regular "force" not changing the direction of action when crossing the trap region.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.45.-a Nonlinear dynamics and nonlinear dynamical systems – 05.70.Fh Phase transitions: general studies

1 Introduction

The ability of noise to produce order in systems, in particular, to induce phase transitions is well established (see, e.g., Refs. [1,2]). Such a phase transition manifests itself in the phase-space density of the system changing its structure, for example, the number of maxima. Noised-induced phase transitions are distinguished from the classical ones by the fact that their cause is not only the features of regular "forces" but also the action of random Langevin "forces". As a result, in particular, the maxima of the distribution function describing the noise-induced phases are not necessarily related to the zero value of the regular "forces".

System of elements with motivated behavior, e.g., fish and bird swarms, car ensembles on highways, stock markets, *etc.* often display noise-induced phase transitions. The formation of a new phase is caused by noise action (for a review see Ref. [3]). The theory of these phenomena is far from being developed well.

The present paper considers a certain class of such systems whose dynamics can be described by two variables x, v which perform a damped harmonic oscillation near the equilibrium point $\{x = 0, v = 0\}$. However, the system "cost" of deviation from the equilibrium can differ substantially for these variables. For example, in driving a

car the control over the relative velocity v is of prime importance in comparison with the correction of the headway distance x. So, under normal conditions a driver should eliminate the relative velocity between her car and a car ahead first and only then correct the headway. In markets the deviation from the supply-demand equilibrium reflecting in price changes also has to exhibit faster time variations than, e.g., the production cost determined by technology capabilities. In physical systems this situation can be also met, e.g., in Pd-metal alloys charged with hydrogen where the structure relaxation exhibits non-monotonic dynamics [4,5]. In these alloys hydrogen atoms and nonequilibrium vacancies form long lived complexes affecting essentially the structure relaxation. Their generation and disappearance governed, in turn, by the structure evolution causes the non-monotonic dynamics which can be described in terms of dynamical traps.

These observations lead to the concept of dynamical traps, a certain "low" dimensional region in the phase space where the main kinetic coefficients specifying the characteristic time scales of the system dynamics become sufficiently large in comparison with their values outside the trap region [6,7]. A trap region is not necessarily to be bounded in all the dimensions and, in this case, it itself cannot lead to the formation of new phases. Moreover, the equilibrium point can be absolutely stable, so without noise there is only one phase in the system. The present Letter analyzes the effect of noise on such a system and

[a] e-mail: ialub@fpl.gpi.ru

demonstrates that additive noise in a system with dynamical traps is able to give rise to new phases. It should be noted that in most models the phase transitions are caused by multiplicative noise, however, additive noise also can give rise to these phenomena [8–10].

2 Noised oscillatory system with dynamical traps

By way of example, the following system typically used to describe the harmonic oscillatory dynamics is considered

$$\frac{dx}{dt} = v, \tag{1}$$

$$\frac{dv}{dt} = -\omega_0^2 \Omega(v) \left[x + \frac{\sigma}{\omega_0} v \right] + \epsilon_0 \xi_v(t). \tag{2}$$

Here x and v are the dynamical variables usually treated as a coordinate and velocity of a certain particle, ω_0 is the circular frequency of oscillations provided the system is not affected by other factors, σ is the damping decrement, and the term $\epsilon_0 \xi_v(t)$ in equation (2) is a random Langevin "force" of intensity ϵ_0 proportional to the white noise $\xi_v(t)$,

$$\langle \xi_v(t) \rangle = 0, \quad \langle \xi_v(t) \xi_v(t') \rangle = \delta(t - t'), \tag{3}$$

with unit amplitude. The function $\Omega(v)$ describes the dynamical trap effect arising in the vicinity of zero value velocity. For this function, the following simple ansatz can be used

$$\Omega(v) = \frac{v^2 + \triangle^2 \vartheta_t^2}{v^2 + \vartheta_t^2}, \tag{4}$$

where the parameter ϑ_t characterizes the thickness of the trap region and the parameter $\triangle \leq 1$ measures the trapping efficacy. When $\triangle = 1$ the dynamical trap effect is ignorable, for $\triangle = 0$ it is most effective.

The characteristic features of the given system are illustrated in Figure 1. The shadowed domain shows the trap region where the regular "force", the former term in equation (2), is depressed. The latter is described by the factor $\Omega(v)$ taking small values in the trap region (for $\triangle \ll 1$). Inside the trap region the system is mainly governed by the random Langevin "force". Outside the trap region it is approximately harmonic.

In order to analyze the system dynamics a dimensionless time t and the dynamical variables η and u are used. Namely, the time t is measured in units of $1/\omega_0$, i.e., $t \to t/\omega_0$ and the units of the coordinate x and the velocity v are ϑ_t/ω_0 and ϑ_t, respectively. So, by introducing the new variables

$$\eta = \frac{x \omega_0}{\vartheta_t} \quad \text{and} \quad u = \frac{v}{\vartheta_t}$$

the dynamical equations (1, 2) read (for the dimensionless time t)

$$\frac{d\eta}{dt} = u, \quad \frac{du}{dt} = -\Omega[u] (\eta + \sigma u) + \epsilon \xi(t), \tag{5}$$
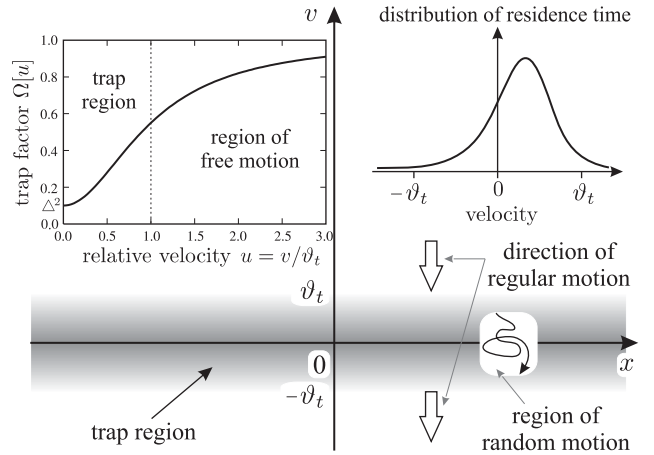

Fig. 1. Characteristic structure of the phase space $\{x, v\}$. The shadowed domain represents the trap region where the regular "force" is depressed and the system motion is random. The regular "force" depression is described by the factor $\Omega(v)$ illustrated in the left window. The essence of the trap effect on the system dynamics is shown in the right window. Outside the trap region the system dynamics is mainly regular.

where the noise $\xi(t)$ obeys conditions like equalities (3), the parameter $\epsilon = \epsilon_0/(\sqrt{\omega_0} \vartheta_t)$, and the function

$$\Omega[u] = \frac{u^2 + \triangle^2}{u^2 + 1}.$$

Without noise, this system has only one stationary point $\{\eta = 0, u = 0\}$ being stable because it possesses a Liapunov function

$$\mathcal{H}(\eta, u) = \frac{\eta^2}{2} + \frac{u^2}{2} + \frac{1 - \triangle^2}{2} \ln \left(\frac{u^2 + \triangle^2}{\triangle^2} \right). \tag{6}$$

This Liapunov function attains the absolute minimum at the point $\{\eta = 0, u = 0\}$ and obeys the inequality

$$\frac{d\mathcal{H}(\eta, u)}{dt} = -\sigma u^2 < 0 \quad \text{for} \quad u \neq 0. \tag{7}$$

In particular, if $\sigma = 0$ and $\epsilon = 0$ then function (6) is the first integral of the system. In what follows, the values σ and ϵ will be treated as small parameters.

The present paper demonstrates the fact that the noise $\xi(t)$ can cause a phase transition in the given system. It manifests itself in that the distribution function $\mathcal{P}(\eta, u)$ changes form from a unimodal to a bimodal one. The dynamics of system (5) was analyzed numerically using a high order stochastic Runge-Kutta method [11] (see also Ref. [12]). The distribution function $\mathcal{P}(\eta, u)$ was calculated numerically by finding the cumulative time during which the system is located inside a given mesh on the (η, u)-plane for a path of a sufficiently long time of motion, $t \approx 500,000$. The size of mesh was chosen to be about 1% of the dimension characterizing the system location on the (η, u)-plane.

The evolution of the distribution function $\mathcal{P}(\eta, u)$ is shown in Figure 2 in the form of the level contours dividing
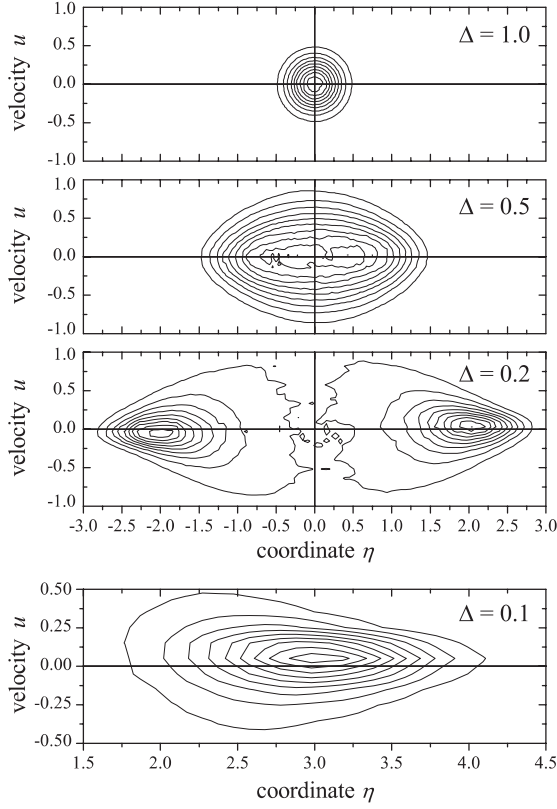
Fig. 2. Evolution of the distribution function $\mathcal{P}(\eta, u)$ (shown by level contours) as the parameter $\triangle$ decreases. In numerical calculations the values $\sigma = 0.1$ and $\epsilon = 0.1$ were used. The lower window depicts only one maximum of the distribution function.

Fig. 3. The form of the distribution function $\mathcal{P}(\eta, u)$ for the parameters $\sigma = 0.1$, $\epsilon = 0.1$, and $\triangle = 0.2$.

Fig. 4. A typical fragment of the system path going through the trap region. The parameters $\sigma = 0.1$, $\epsilon = 0.1$, and $\triangle = 0.01$ were used in numerical simulations in order to make the trap effect more pronounced.

the variation scale into ten equal parts. The upper window corresponds to the case of $\triangle = 1$ where the trap effect is absent and the distribution function is unimodal. The third window illustrates the case when the distribution function has the well pronounced bimodal shape shown also in Figure 3. Comparing the three upper windows in Figure 2 it becomes evident that there is a certain relation $\Phi_c(\triangle, \sigma, \epsilon) = 0$ between the parameters $\triangle$, σ, and ϵ when the system undergoes a second order phase transition which manifests itself in the change of the shape of the phase space density $\mathcal{P}(\eta, u)$ from unimodal to bimodal. In particular, for $\sigma = 0.1$ and $\epsilon = 0.1$ the critical value of the parameter $\triangle$ is $\triangle_c(\sigma, \epsilon) \approx 0.5$ as seen in the second window.

3 Mechanism of the phase transition

To understand the mechanism of the noised induced phase transition observed numerically in the given system, consider a typical fragment of the system motion through the trap region for $\triangle \ll 1$ that is shown in Figure 4. When it goes into the trap region $\mathcal{Q}_t$, $-\vartheta_t \ll v \ll \vartheta_t$, the regular "force" $\Omega[u](\eta + \sigma u)$ containing the trap factor $\Omega[u]$ and governing the regular motion becomes small. So inside this region the system dynamics becomes random due
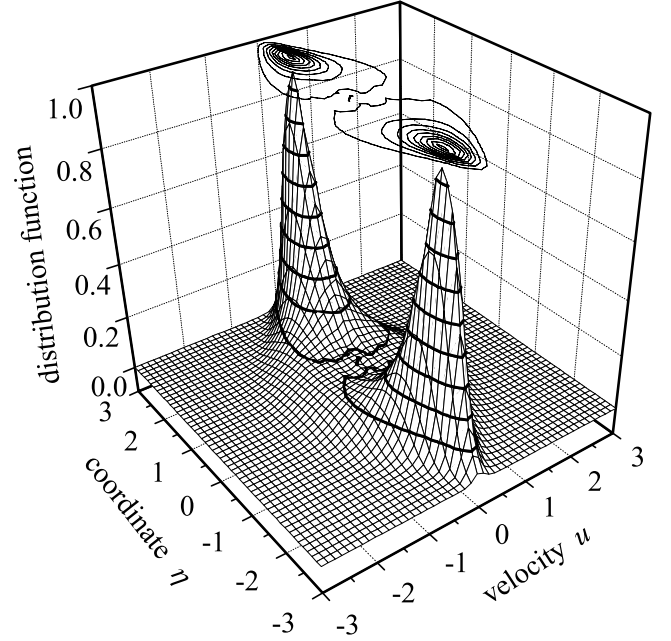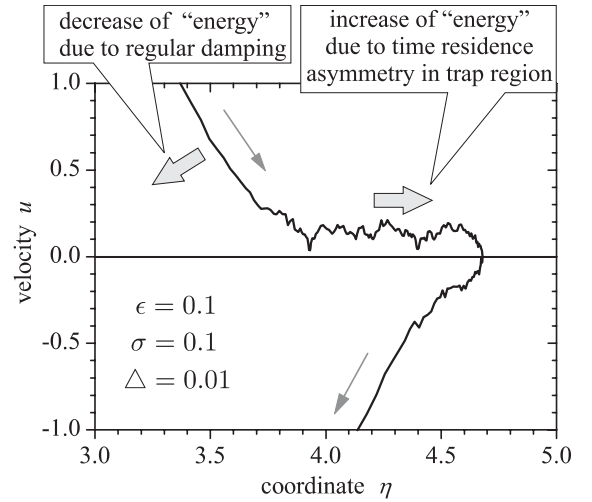
to the remaining weak Langevin "force" $\epsilon\xi(t)$. However, the boundaries $\partial_+\mathcal{Q}_t$ (where $v \sim \vartheta_t$) and $\partial_-\mathcal{Q}_t$ (where $v \sim -\vartheta_t$) are not identical in properties with respect to the system motion. At the boundary $\partial_+\mathcal{Q}_t$ the regular "force" leads the system inwards the trap region $\mathcal{Q}_t$, whereas at the boundary $\partial_-\mathcal{Q}_t$ it causes the system to leave the region $\mathcal{Q}_t$. Outside the trap region $\mathcal{Q}_t$ the regular "force" is dominant. Thereby, from the standpoint of the system motion inside the region $\mathcal{Q}_t$, the boundary $\partial_+\mathcal{Q}_t$ is "reflecting" whereas the boundary $\partial_-\mathcal{Q}_t$ is "absorbing".

As a result the distribution of the residence time at different points of the region $\mathcal{Q}_t$ should be asymmetric, as

schematically shown in Figure 1 (the right window). This asymmetry is also seen in the distribution function $\mathcal{P}(\eta, u)$ obtained numerically. Its maxima are located at the points with non-zero values of the velocity, which is clearly visible in the lower window of Figure 2. Therefore, during location inside the trap region the mean velocity of the system must be positive and it tends to go away from the origin. This effect gives rise to an increase in the "energy" $\mathcal{H}(\eta, u)$. Outside the trap region the "energy" $\mathcal{H}(\eta, u)$ decreases according to expression (7). So, when the former effect becomes sufficiently strong, i.e., the random "force" intensity ϵ exceeds a certain critical value, $\epsilon > \epsilon_c(\triangle, \sigma)$, the distribution function $\mathcal{P}(\eta, u)$ becomes bimodal.

The system location with respect to the velocity v is due to the regular "force" being sufficiently strong outside the trap region, so the system spends the main time inside this region. Its location with respect to the coordinate x is caused by the fact that the region where the Langevin "force" mainly affects the system dynamics decreases in thickness as the coordinate x increases. The latter tendency takes place because the regular "force", the first term in equation (2), is proportional to x. In fact, the thickness $U(\eta)$ of the trap region in the vicinity of the point $\{\eta, u = 0\}$ can be estimated using the condition of the equality of the characteristic times, $\tilde{t}_s$ and $\tilde{t}_d$, during which the system crosses the trap region under action of the regular "force" and the random Langevin "force". So

$$\tilde{t}_s \sim \frac{U}{\Omega[U]\eta} \sim \tilde{t}_d \sim \frac{U^2}{\epsilon^2}.$$

and setting for the sake of simplicity $\triangle = 0$ we get the estimate

$$U(\eta) \sim \epsilon^{2/3} \eta^{-1/3}.$$

Moreover, let the mean velocity in the trap region caused by the residence time asymmetry be about $U(\eta)$. Then the characteristic increase $\delta\eta$ of the coordinate η got by the system when crossing the trap region is estimated as $\delta\eta \sim U\tilde{t}_s \sim 1/\eta$. Thereby the dynamical trap effect becomes weaker as the "energy" $\mathcal{H}(\eta, u)$ increases. By contrast, according to equation (7) the higher the "energy", the stronger its dissipation caused by the regular "force".

4 Conclusion

The present paper has considered noise-induced phase transitions in systems with dynamical traps. By way of example, a simple oscillatory system $\{x, v = \dot{x}\}$ is studied when the trap region is located in the vicinity of the x-axis and without noise the stationary

point $\{x = 0, v = 0\}$ is absolutely stable. For this system as shown numerically the additive white noise can cause the phase-space density to take the bimodal shape.

In contrast to the classical phase transitions the position of the new noise-induced phases is not specified by the zero-values of regular "forces" even approximately. The cause of the observed phase transition is the asymmetry of the residence time distribution inside the trap region. This asymmetry is due the cooperative effect of the regular "force" outside the trap region and the random Langevin "force" inside it. The regular "force" does not change the direction when crossing the trap region, inside this region it is depressed only. As a result, for the motion inside the trap region one of its boundaries is "reflecting", whereas the other is "absorbing", which induces the residence time asymmetry. The latter gives rise to increase in the system "energy". Outside the trap region the regular "force" causes the "energy" to decrease.

References

1. W. Horsthemke, R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984)
2. *Stochastic Dynamics*, Lecture Notes in Physics, Vol. 484, edited by L. Schimansky-Geier, T. Pöschel (Springer-Verlag, Berlin, 1998)
3. D. Helbing, Rev. Mod. Phys. **73**, 1067 (2001)
4. A.A. Katsnelson, A.I. Olemskoi, I.V. Suhorukova, G.P. Revkevich, Physics-Uspekhi **165**, 331 (1995)
5. V.M. Avdjukhina, A.A. Anishchenko, A.A. Katsnelson, G.P. Revkevich, Perspekt. Materials **4**, 5 (2002) (in Russian)
6. I.A. Lubashevsky, V.V. Gafiychuk, A.V. Demchuk, Physica A **255**, 406 (1998)
7. I. Lubashevsky, R. Mahnke, P. Wagner, S. Kalenkov, Phys. Rev. E **66**, 016117 (2002)
8. A.A. Zaikin, L. Schimansky-Geier, Phys. Rev. E **58**, 4355 (1998)
9. P.S. Landa, A.A. Zaikin, L. Schimansky-Geier, Chaos, Solitons, Fractals **9**, 1367 (1998)
10. A. Zaikin, J. Garcia-Ojalvo, L. Schimansky-Geier, Phys. Rev. E **60**, R6275 (1999)
11. K. Burrage, P.M. Burrage, App. Num. Math. **22**, 81 (1996)
12. P.M. Burrage, *Numerical methods for stochastic differential equations*, Ph.D. Thesis (University of Queensland, Brisbane, Queensland, Australia, 1999)